# METU - NCC

| Differential Equations<br>Midterm 2 | |
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| Code : *MAT 219*<br>Acad.Year: *2015-2016*<br>Semester : *Fall*<br>Date : *13.12.2015*<br>Time : *9:40*<br>Duration : *110 min* | Last Name: List #:<br>Name :<br>Student # :<br>Signature :<br>Exam length: 5 PAGES<br>Total: 100 POINTS |

| 1. (24) | 2. (28) | 3. (17) | 4. (14) | 5. (17) | |
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Please draw a $\boxed{\text{box}}$ around your answers. No calculators, cell-phones, notes, etc. allowed.

**1.1.** *(10+14=24pts)* Find the general solutions for the following systems.

(A) $\vec{x}\,' = \begin{bmatrix} 1 & 2 \\ -5 & 3 \end{bmatrix} \vec{x}$

Eigenvalues $\det\begin{bmatrix} 1-\lambda & 2 \\ -5 & 3-\lambda \end{bmatrix} = 0$

$\lambda^2 - 4\lambda + 13 = 0$

$(\lambda-2)^2 + 9 = 0$

$\lambda = 2 \pm 3i$

Eigenvectors $\begin{bmatrix} 1-(2+3i) & 2 \\ -5 & 3-(2+3i) \end{bmatrix} \underline{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} -1-3i & 2 \\ -5 & 1-3i \end{bmatrix} \underline{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\underline{v} = \begin{bmatrix} 2 \\ 1+3i \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \end{bmatrix} i$

Solution: $\underline{x} = c_1\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\cos 3t - \begin{bmatrix} 0 \\ 3 \end{bmatrix}\sin 3t\right)e^{2t} + c_2\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\sin 3t + \begin{bmatrix} 0 \\ 3 \end{bmatrix}\cos 3t\right)e^{2t}$

or $\underline{x} = \begin{bmatrix} 2e^{2t}\cos 3t & 2e^{2t}\sin 3t \\ e^{2t}\cos 3t - 3e^{2t}\sin 3t & e^{2t}\sin 2t + 3e^{2t}\cos 2t \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

(B) $\vec{x}\,' = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} \vec{x} + \begin{bmatrix} 0 \\ 2e^{2t} \end{bmatrix}$

Eigenvalues:

$\lambda^2 - 4\lambda + 4 = 0$

$\lambda = 2, 2$

Eigenvectors:

$\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \underline{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightsquigarrow v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

gen. $\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \underline{w} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \rightsquigarrow \underline{w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ (or $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$)

$\Psi = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} e^{2t}$

$\Psi^{-1} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{-1} e^{-2t} = \begin{bmatrix} 1 & -t \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} e^{-2t}$

$\int \Psi^{-1} g\, dt = \int \begin{bmatrix} 1 & -t \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} e^{-2t} \begin{bmatrix} 0 \\ 2e^{2t} \end{bmatrix} dt = \int \begin{bmatrix} 1 & -t \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix} dt$

$= \int \begin{bmatrix} 1 & -t \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix} dt = \int \begin{bmatrix} -2t \\ 2 \end{bmatrix} dt = \begin{bmatrix} -t^2 \\ 2t \end{bmatrix}$

$\underline{x} = c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{2t} + c_2\begin{bmatrix} t \\ t+1 \end{bmatrix}e^{2t} + \boxed{\left(-t^2\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{2t} + 2t\begin{bmatrix} t \\ t+1 \end{bmatrix}e^{2t}\right)}$ simplifies to $\begin{bmatrix} t^2 \\ t^2+2t \end{bmatrix}e^{2t}$

**2.1.** *(12+4=16pts)* Solve the initial value problem $\vec{x}\,' = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 3 & 0 \\ -2 & 5 & -1 \end{bmatrix} \vec{x}$ with $\vec{x}(0) = \begin{bmatrix} 9 \\ 0 \\ -13 \end{bmatrix}$.

$\Delta(\lambda) = \begin{vmatrix} 1-\lambda & -1 & 0 \\ 1 & 3-\lambda & 0 \\ -2 & 5 & -1-\lambda \end{vmatrix} = -(\lambda+1)(\lambda-2)^2$, $\sigma(A) = \{2^{(2)}, -1^{(1)}\}$. For $\lambda = 2$ we have

$A - 2 = \begin{bmatrix} -1 & -1 & 0 \\ 1 & 1 & 0 \\ -2 & 5 & -3 \end{bmatrix}$, $(A-2)^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 13 & -8 & 9 \end{bmatrix}$; $V_{2,1} = \{x+y=0, -2x+5y=3z\} \subsetneq V_{2,2} = \{13x - 8y + 9z = 0\}$

$\vec{f}_1 = \begin{bmatrix} 9 \\ 0 \\ -13 \end{bmatrix}$, $\vec{f}_2 = (A-2)\vec{f}_1 = \begin{bmatrix} -9 \\ 9 \\ 21 \end{bmatrix}$. For $\lambda = -1$, $A + 1 = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 4 & 0 \\ 0 & 4 & 0 \end{bmatrix}$, $V_{-1} = \{x = y = 0\}$. Thus

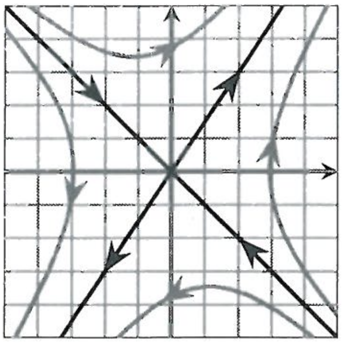
$P = \begin{bmatrix} 9 & -9 & 0 \\ 0 & 9 & 0 \\ -13 & 21 & 1 \end{bmatrix}$, $J = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$, $e^{Jt} = \begin{bmatrix} e^{2t} & 0 & 0 \\ te^{2t} & e^{2t} & 0 \\ 0 & 0 & e^{-t} \end{bmatrix}$. Consequently

$$\Psi(t) = Pe^{Jt} = \begin{bmatrix} 9 & -9 & 0 \\ 0 & 9 & 0 \\ -13 & 21 & 1 \end{bmatrix} \begin{bmatrix} e^{2t} & 0 & 0 \\ te^{2t} & e^{2t} & 0 \\ 0 & 0 & e^{-t} \end{bmatrix} = \begin{bmatrix} 9(1-t)e^{2t} & -9e^{2t} & 0 \\ 9te^{2t} & 9e^{2t} & 0 \\ (-13+21t)e^{2t} & 21e^{2t} & e^{-t} \end{bmatrix}$$

What is $\lim\limits_{t \to \pm\infty} \vec{x} = ?$

$\begin{bmatrix} 9 \\ 0 \\ -13 \end{bmatrix} = \Psi(0) \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \Rightarrow c_1 = 1, c_2 = c_3 = 0 \Rightarrow \vec{x}(t) = \begin{bmatrix} 9(1-t)e^{2t} \\ 9te^{2t} \\ (-13+21t)e^{2t} \end{bmatrix} \Rightarrow$ $\lim\limits_{t \to +\infty} \|\vec{x}(t)\| = \infty$, $\lim\limits_{t \to -\infty} \|\vec{x}(t)\| = 0.$

**2.2.** *(6pts)* Some phase plane solutions for a $2\times 2$ system $\vec{x}\,' = A\vec{x}$ are given below. Describe the eigenvalues and eigenvectors of $A$.



$\lambda_1 < 0$ $\qquad$ $\lambda_2 > 0$

$\vec{f}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\qquad$ $\vec{f}_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

**2.3.** *(3+3=6pts)* Consider the vector functions $\vec{u} = \begin{bmatrix} e^t \\ -e^{2t} \\ 0 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} e^{-2t} \\ e^t \\ 0 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} t \\ 0 \\ \sin(t) \end{bmatrix}$.

(A) Is it possible for $\{\vec{u}, \vec{v}, \vec{w}\}$ to be a complete set of fundamental solutions for a $3\times 3$ system $\vec{x}\,' = \mathbf{P}\vec{x}$ where $\mathbf{P} = \mathbf{P}(t)$ is a continuous matrix valued function on $\mathbb{R}$? Explain.
(Hint: Could these solutions solve initial value problems at all $t_0 \in \mathbb{R}$?)

It is impossible, for $W(t) = \begin{vmatrix} e^t & e^{-2t} & t \\ -e^{2t} & e^t & 0 \\ 0 & 0 & \sin(t) \end{vmatrix} = (e^{2t}+1)\sin(t)$

and $W(\pi k) = 0$, $W(\pi/2) \neq 0$.

(B) Could $\{\vec{u}, \vec{v}\}$ be a complete set of fundamental solutions for some $3\times 3$ system $\vec{x}\,' = \mathbf{P}\vec{x}$? Explain. No, the dimension of Sol is supposed to be 3.

Name: ID:

**3.1.** *(3×3=9pts)* Find the general solution for the following linear constant coefficient equations.

(A) $y'' - 8y' + 15y = 0$.

$\lambda^2 - 8\lambda + 15 = (\lambda-3)(\lambda-5) \Rightarrow \sigma(A) = \{3, 5\}$ for $A = \begin{bmatrix} 0 & 1 \\ -15 & 8 \end{bmatrix}$. So, $y = C_1 e^{3t} + C_2 e^{5t}$

(B) $y^{(5)} - 4y^{(3)} = 0$.

$\lambda^5 - 4\lambda^3 = \lambda^3(\lambda^2 - 4) = \lambda^3(\lambda-2)(\lambda+2) \Rightarrow \sigma(A) = \{0^{③}, 2, -2\}$

$y = C_1 + C_2 t + C_3 t^2 + C_4 e^{2t} + C_5 e^{-2t}$

(C) A constant coefficient, linear, homogeneous differential equation has the characteristic equation $r(r^2 - 6r + 25)^3 = 0$. What is the general solution to the differential equation? $\sigma(A) = \{0, 3 \pm 4i^{③}\}$

$y = C_1 + C_2 e^{3t}\cos(4t) + C_3 e^{3t}\sin(4t) + C_4 te^{3t}\cos(4t) + C_5 te^{3t}\sin(4t) +$
$+ C_6 t^2 e^{3t}\cos(4t) + C_7 t^2 e^{3t}\sin(4t)$.

**3.2.** *(6+2=8pts)* The following parts are about the nonhomogeneous diff. eqn. $y'' - 6y' + 9y = 2e^{3t}$.

(A) Find a **particular solution** $Y_p$ to the differential equation using **variation of parameters**.

$\lambda^2 - 6\lambda + 9 = (\lambda-3)^2 \Rightarrow y_h = C_1 te^{3t} + C_2 e^{3t}$, $\Psi(t) = \begin{bmatrix} te^{3t} & e^{3t} \\ (1+3t)e^{3t} & 3e^{3t} \end{bmatrix}$

Put $\vec{Y}(t) = \Psi(t)\vec{C}(t)$. Then $\vec{C}'(t) = \Psi(t)^{-1}\vec{b}(t) = -e^{-6t}\begin{bmatrix} 3e^{3t} & -e^{3t} \\ -(1+3t)e^{3t} & te^{3t} \end{bmatrix}\begin{bmatrix} 0 \\ 2e^{3t} \end{bmatrix}$

$= \begin{bmatrix} -3 & 1 \\ 1+3t & -t \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -2t \end{bmatrix} \Rightarrow \vec{C}(t) = \begin{bmatrix} 2t \\ -t^2 \end{bmatrix}$. It follows that

$\vec{Y}(t) = \begin{bmatrix} te^{3t} & e^{3t} \\ (1+3t)e^{3t} & 3e^{3t} \end{bmatrix}\begin{bmatrix} 2t \\ -t^2 \end{bmatrix} = \begin{bmatrix} 2t^2e^{3t} - t^2e^{3t} \\ * \end{bmatrix} = \begin{bmatrix} t^2e^{3t} \\ * \end{bmatrix}$,

that is, $Y(t) = t^2 e^{3t}$ is a special solution to nonhomg. dif. eq.

(B) Write the general solution to the differential equation.

$y = C_1 te^{3t} + C_2 e^{3t} + t^2 e^{3t}$.

**4.1.** *(8+6=14pts)* The following parts are both about **undetermined coefficients.**

(A) Find a particular solution to $y''' + y'' + y' + y = 2e^{-t} + 4t$ using **undetermined coefficients**.
(*Hint: The characteristic equation is* $(r+1)(r^2+1) = 0$.)

$y_h = C_1 e^{-t} + C_2 \cos(t) + C_3 \sin(t)$, $Y(t) = tAe^{-t} + Bt + C$.

Then $Y'(t) = -Ate^{-t} + Ae^{-t} + B$, $Y''(t) = Ate^{-t} - 2Ae^{-t}$,

$Y'''(t) = -Ate^{-t} + 3Ae^{-t} \Rightarrow$

$-Ate^{-t} + Ate^{-t} - Ate^{-t} + tAe^{-t} + 3Ae^{-t} - 2Ae^{-t} + Ae^{-t} + Bt + B + C =$

$= 2e^{-t} + 4t$ or $2Ae^{-t} + Bt + B + C = 2e^{-t} + 4t \Rightarrow$

$A = 1$, $B = 4$, $C = -B = -4$, that is,

$Y(t) = te^{-t} + 4t - 4$ is a special solution.

(B) A homogeneous equation $y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = 0$ has fundamental solutions

$$1,\ t,\ t^2,\ e^{2t},\ \sin(4t),\ \cos(4t),\ t\sin(4t),\ t\cos(4t).$$

(i) What is **n** (the **order** of the differential equation)?

$n = 8$

(ii) What is the **form** of the particular solution $Y_p$ used to solve

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1}y' + a_n y = t^3 e^{2t} + \cos(5t) + t^2 e^{6t}\sin(7t)$$

in the method of undetermined coefficients?

(DO NOT SOLVE FOR THE COEFFICENTS!)

$Y(t) = t(A_1t^3 + A_2t^2 + A_3t + A_4)e^{2t} + B_1\cos(5t) + B_2\sin(5t) +$

$+ (C_1t^2 + C_2t + C_3)e^{6t}\cos(7t) + (D_1t^2 + D_2t + D_3)e^{6t}\sin(7t)$

**5.1.** *(7+3=10pts)* The following parts are about the differential equation $t^2\, y''' - 2t\, y'' + 2y' = 0$.

(A) Convert the third order differential equation into a first order linear 3×3 system.

$$y''' = \frac{2}{t^2} y' + \frac{2t}{t^2} y''$$

$$x_3' = (y'')' = \frac{2}{t^2} x_2 + \frac{2}{t} x_3$$

$$\boxed{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 2/t^2 & 2/t \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}$$

(B) The general solution to the third order differential equation is $y = c_1 + c_2\, t^2 + c_3\, t^3$.
What is the general solution to the 3×3 system from (A)?

$$\left.\begin{aligned} x_1 &= y = c_1 + c_2 \cdot t^2 + c_3 \cdot t^3 \\ x_2 &= y' = c_2 \cdot 2t + c_3 \cdot 3t^2 \\ x_3 &= y'' = c_2 \cdot 2 + c_3 \cdot 6t \end{aligned}\right\} \quad \boxed{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} t^2 \\ 2t \\ 2 \end{bmatrix} + c_3 \begin{bmatrix} t^3 \\ 3t^2 \\ 6t \end{bmatrix}}$$

**5.2.** *(7pts)* The following second order differential equation becomes linear, constant coefficient after substituting $y = tu$.

$$t^2\, y'' - 2t\, y' + (t^2 + 2)\, y = 0 \qquad (t \neq 0)$$

Substitute $y = tu$ and solve for $u$. Write the solution function $y$.

$$\left.\begin{aligned} y &= tu \\ y' &= u + tu' \\ y'' &= 2u' + tu'' \end{aligned}\right\}$$

$$(t^2+2)y = (t^2+2)tu$$

$$-2t\, y' = (-2t)u + (-2t)tu'$$

$$t^2 y'' = (t^2)2u' + (t^2)tu''$$

$$0 = \cancel{t^3}u + 0 + \cancel{t^3}u''$$

$t\ (t \neq 0)$

$$0 = u + u''$$

$0 = 1 + r^2$ is char. eqn.

$r = \pm i$

$$u = c_1 \cos t + c_2 \sin t$$

$$\boxed{y = t \cdot u = c_1 (t\cos t) + c_2 (t \sin t)}$$